

Stochastic Block Models and Reconstruction

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March 21, 2012

Abstract

Consider the following *stochastic block model* of a random graph consisting of two clusters of size approximately $n/2$. The cross-class edge probability is a/n and the within-class probability is b/n . Decelle et al. conjectured a threshold for the algorithmic problem of reconstructing the hidden labels in a way that is correlated with the true partition. Their conjecture is that the threshold is $(a - b)^2 = 2(a + b)$ which is exactly the threshold for the corresponding reconstruction problem on trees.

We prove one side of this conjecture, i.e., that reconstruction is impossible when $(a - b)^2 \leq 2(a + b)$. Moreover, we show that the stochastic block model is contiguous to an Erdős-Renyi model when $(a - b)^2 < 2(a + b)$ and orthogonal to it when $(a - b)^2 > 2(a + b)$.

1 Introduction

1.1 Stochastic Block Models

The study of random networks has seen a surge of interest, driven partly by social and biological applications. Random networks that exhibit a “community” or “cluster” structure are of particular interest and many such models are now being studied; a discussion of similar models can be found in [20]. In these models, a collection of vertices are divided into several classes and then a random graph is drawn in some way that depends on the class membership. In many models, such random graphs tend to feature

^{*}Supported by NSF grant DMS-1106999 and DOD ONR grant N000141110140

many connections between vertices of the same class and fewer connections between vertices of different classes.

Perhaps the simplest random graph model with a community structure is the stochastic block model, in which vertices are assigned independent random labels and then the edges are chosen independently, with the probability of an edge appearing depending only on the label of its endpoints. The stochastic block model and its relatives have been studied since the 1980’s, with literature extending across physics, statistics and computer science. It was introduced by Holland et al. [16], while a similar model was proposed independently by Bui et al. [5].

The simplest of these models may be defined by first assigning each vertex to one of the two clusters independently and uniformly. Then each potential edge (u, v) of the graph is included independently with probability a/n if u, v belong to the same cluster and b/n if they belong to different clusters. It is this model that we will discuss in the current paper.

In physics and statistics, such models were motivated by physical, biological, and social systems – see [13] for a survey discussing these motivations. In computer science, on the other hand, the study of such models was driven by the study of the graph bisection problem, which is computationally hard in the worst case [14], but is easy on average under the stochastic block model with suitable parameters. We should note that the stochastic block model is usually called the “planted partition model” in the computer science literature.

1.2 Block Reconstruction

In all areas, the most prominent question is that of “community reconstruction.” Given a random graph with the vertex labels erased, is it possible to reconstruct the vertex labels just by looking at the graph structure? This question has been studied many [1, 4, 6, 8, 11, 17, 18, 21, 25, 26] times, with many different methods. For example the results of [11] (following similar results for a slightly different model) imply that if $a - b \geq n^{1/4+\epsilon}$ then the true partition can be found with high probability. The results of [21] imply that the true partition can be found with probability $1 - \delta$ if $(a - b)^2/a > C \log(n/\delta)$ where C is a large constant (see [1, 25] for related results).

Each of the above works contains an algorithm that will exactly reconstruct the true vertex labels with high probability under certain model parameters.

When the goal is perfect recovery of all the labels, the community reconstruction question only makes sense for random graphs with degrees tending

to infinity. A random sparse graph on n nodes with average degree less than some small constant times $\log n$ will contain many isolated vertices whose labels clearly cannot be inferred. Even when looking at the connected component it is easy to see that many vertices with small degrees will be inferred incorrectly when $a, b > 0$.

1.3 Sparse Block Models and Our Contribution

For sparse graphs it is natural to propose a relaxed problem in which one only seeks to find a labelling that is positively correlated with the true labelling. This problem was recently studied by Coja-Oghlan [7] who proved that this can be done for suitable choices of the model parameters. In particular, in the setting of the current paper this can be done when $(a - b)^2 > C(a + b)$ for some large constant C .

More recently, Decelle et al. [9, 10] conjectured, based on deep but non-rigorous ideas from statistical physics, a threshold that separates the parameters for which reconstruction is possible from the parameters for which it is not. Their conjecture is that it is possible to reconstruct a correlated partition if $(a - b)^2 > 2(a + b)$, but that it is not possible if $(a - b)^2 < 2(a + b)$. We will rigorously prove some of the ideas from their paper.

The conjecture of [9] relates the block reconstruction problem to the tree reconstruction problem, see e.g [22]. Consider the following multi-type branching process where there are two types of particles named \pm . Each particle gives birth to Poisson with parameter a particles of the same type and a Poisson with parameter b particles of the complementary type. In the reconstruction problem, the goal is to recover the label of the root of the tree from the labels of level r where $r \rightarrow \infty$.

Results of Kesten and Stigum for multi-type branching processes [19] imply that if $(a - b)^2 > 2(a + b)$ then it is possible to recover the root value with non-trivial probability and results of Evans et al. imply that if $(a - b)^2 \leq 2(a + b)$ then it is impossible to recover the root with non-trivial probability.

It is not hard to imagine that there is a connection between the two problems. If we look at a neighborhood of a vertex in the random graph model, it is a tree with high probability and the tree distribution is given in the limit by the multi-type branching process defined above.

As part of this work, we make this connection rigorous. By doing so, we prove one side of [9]’s threshold conjecture: we show that it is impossible to recover the labels below the threshold – when $(a - b)^2 \leq 2(a + b)$. Then, we show that the conjectured threshold is a threshold for a different question: below the threshold, a graph from the stochastic block model is

almost indistinguishable from an Erdős-Renyi random graph, but above the threshold they can be distinguished with probability tending to 1.

2 The model and the main results

The stochastic block model $\mathcal{G}(n, p, q)$ is a model for random, $\{\pm\}$ -labelled graphs on n nodes – more generally, there can be more than two labels but we will only consider the two-label case here. It is easiest to describe this model by saying how to sample from it: to generate a pair $(G, \sigma) \sim \mathcal{G}(n, p, q)$, we first choose a uniformly random labelling $\sigma \in \{\pm\}^n$. Then for every pair $\{u, v\}$ independently, we draw an edge between u and v with probability p if $\sigma_u = \sigma_v$ and with probability q otherwise. Of course, when $p = q$ then the model $\mathcal{G}(n, p, q)$ is the same as the Erdős-Renyi model $\mathcal{G}(n, p)$.

Equivalently, we can specify $\mathcal{G}(n, p, q)$ by writing down its probability mass function: let

$$V_{uv}(G, \sigma) = \begin{cases} p & \text{if } \sigma_u = \sigma_v \text{ and } \{u, v\} \in E(G) \\ q & \text{if } \sigma_u \neq \sigma_v \text{ and } \{u, v\} \in E(G) \\ 1 - p & \text{if } \sigma_u = \sigma_v \text{ and } \{u, v\} \notin E(G) \\ 1 - q & \text{if } \sigma_u \neq \sigma_v \text{ and } \{u, v\} \notin E(G). \end{cases}$$

Then

$$\mathbb{P}(G, \sigma) = 2^{-n} \prod_{\{u, v\}} V_{uv}(G, \sigma).$$

In this article, we will be focused on the sparse case, where both p and q are $O(1/n)$. Therefore, let us introduce two new parameters $a, b > 0$ and set $p = \frac{a}{n}$, $q = \frac{b}{n}$. From now on, \mathbb{P}_n will denote the distribution $\mathcal{G}(n, \frac{a}{n}, \frac{b}{n})$. Although \mathbb{P}_n is a joint distribution on both graphs and labels, we will sometimes write $G \sim \mathbb{P}_n$ when we really mean that G is drawn from the marginal distribution.

Suppose that $a + b > 2$, so that $G \sim \mathbb{P}_n$ has a giant component. Then there is some hope of reconstructing, from the unlabelled graph, a labelling τ that is correlated with the true labels σ in the sense that $\mathbb{P}(\tau_u = \tau_v | \sigma_u = \sigma_v) > \frac{1}{2}$ for any fixed pair u, v of vertices. Decelle et al. [9] conjectured that label reconstruction is possible if $(a - b)^2 > 2(a + b)$ and impossible if $(a - b)^2 < 2(a + b)$. Our first result is a partial answer to their conjecture; specifically, we show that reconstruction is impossible when $(a - b)^2 \leq 2(a + b)$. Note that our result includes the case $(a - b)^2 = 2(a + b)$, for which Decelle et al. did not conjecture any particular behavior.

Theorem 2.1. *If $a + b > 2$ and $(a - b)^2 \leq 2(a + b)$ then, for any fixed vertices u and v ,*

$$\mathbb{P}_n(\sigma_u = + | G, \sigma_v = +) \rightarrow \frac{1}{2} \text{ a.a.s.}$$

In particular, this shows that even an easier problem cannot be solved: if we take two random vertices of G , no algorithm can tell whether or not they have the same label. This is an easier task than label recovery because we no longer ask the algorithm to label all the vertices; we only ask it to say whether two of them have the same label or not.

As for the other half of the conjecture, Coja-Oghlan [7] showed that reconstruction is possible provided that $(a - b)^2 \geq C(a + b)$ for some unspecified constant C . What remains open, therefore, is to show that one can take $C = 2$.

The proof of Theorem 2.1 follows from a connection with Markov processes on trees. It is well-known that a small neighborhood in a sparse Erdős-Renyi graph looks like a Galton-Watson tree. Not surprisingly, this is still true in $\mathcal{G}(n, \frac{a}{n}, \frac{b}{n})$. Moreover, we can show that the labels in a small neighborhood look as though they were obtained by running a Markov process on the tree. After making this connection suitably precise, we can apply the Kesten-Stigum reconstruction threshold, which says that the labels at the leaves of the tree don't tell us anything about the label of the root. From there, it doesn't require a great leap of faith to believe that if the boundary of some neighborhood centered at u gives us no information about σ_u , then the label of some far-off vertex v won't tell us anything either.

Our second result gives a threshold for a different question. So far we have been only considering $\mathbb{P}_n = \mathcal{G}(n, \frac{a}{n}, \frac{b}{n})$. Now let $\mathbb{P}'_n = \mathcal{G}(n, \frac{a+b}{2n})$ be the Erdős-Renyi model that has the same average degree as \mathbb{P}_n . If we were to give you a graph G which was drawn from either \mathbb{P}_n or \mathbb{P}'_n , would you be able to tell which one it came from? It turns out that the answer is “yes” when $(a - b)^2 > 2(a + b)$ and “sometimes” when $(a - b)^2 < 2(a + b)$. The case $(a - b)^2 = 2(a + b)$ remains open.

Theorem 2.2. *If $(a - b)^2 > 2(a + b)$ then \mathbb{P}_n and \mathbb{P}'_n are asymptotically orthogonal. In other words, there exist events A_n such that $\mathbb{P}_n(A_n) \rightarrow 1$ and $\mathbb{P}'_n(A_n) \rightarrow 0$.*

If $(a - b)^2 < 2(a + b)$ then \mathbb{P}_n and \mathbb{P}'_n are mutually contiguous i.e., for a sequence of events A_n , $\mathbb{P}_n(A_n) \rightarrow 0$ if, and only if, $\mathbb{P}'_n(A_n) \rightarrow 0$.

We should emphasize that \mathbb{P}_n and \mathbb{P}'_n do not converge to one another, even below the threshold. In fact, as long as $a \neq b$, one can tell with prob-

ability strictly bigger than $\frac{1}{2}$ whether a given graph came from \mathbb{P}_n or \mathbb{P}'_n . However, the probability of success cannot converge to 1 if $(a-b)^2 < 2(a+b)$.

The two parts of Theorem 2.2 require two separate proofs. The first part is quite straightforward: we show that you can tell which model a graph comes from just by counting the number of short cycles that it has. It's well-known that the number of k -cycles in \mathbb{P}'_n is approximately Poisson-distributed with mean $\frac{1}{k}(\frac{a+b}{2})^k$. We will show that the number of k -cycles in \mathbb{P}_n is approximately Poisson-distributed with mean $\frac{1}{k}((\frac{a+b}{2})^k + (\frac{a-b}{2})^k)$. It's then not hard to see that by taking k to increase slowly with n , we can distinguish between the corresponding Poisson random variables as long as $(a-b)^2 > 2(a+b)$.

For the second part of Theorem 2.2, we will show that the random variables $\frac{\mathbb{P}_n(G)}{\mathbb{P}'_n(G)}$ don't have much mass near 0 or ∞ . Since the margin of \mathbb{P}_n is somewhat complicated to work with, the first step is to enrich the distribution \mathbb{P}'_n by adding random labels. Then we show that the random variables $\frac{\mathbb{P}_n(G,\sigma)}{\mathbb{P}'_n(G,\sigma)}$ don't have mass near 0 or ∞ . This essentially involves estimating a partition function, for which we use the small subgraph conditioning method.

We briefly note that Theorem 2.2 has implications for parameter estimation.

Proposition 2.3. *Consider the problem of inferring the parameters a, b from a single sample. Then there exists a consistent estimator for the parameters a and b from a single sample if $(a-b)^2 > 2(a+b)$. There is no consistent estimator for the parameters a, b if $(a-b)^2 < 2(a+b)$.*

Indeed, our cycle-counting results provide an estimator for a and b which is consistent when $(a-b)^2 > 2(a+b)$. On the other hand, we will show that the second half of Theorem 2.2 implies that no estimator can be consistent when $(a-b)^2 < 2(a+b)$.

3 Counting cycles

The main result of this section is that the number of k -cycles of $G \sim \mathbb{P}_n$ is approximately Poisson-distributed. We will then use this fact to show the first part of Theorem 2.2. Actually, Theorem 2.2 only requires us to calculate the first two moments of the number of k -cycles, but the rest of the moments require essentially no extra work, so we include them for completeness.

Theorem 3.1. *Let $X_{k,n}$ be the number of k -cycles of G , where $G \sim \mathbb{P}_n$. If $k = O(\log^{1/4}(n))$ then*

$$X_{k,n} \xrightarrow{d} \text{Pois} \left(\frac{1}{k2^{k+1}} ((a+b)^k + (a-b)^k) \right).$$

Before we prove this, let us explain how it implies the first part of Theorem 2.2. From now on, we will write X_k instead of $X_{k,n}$.

Proof of the first part of Theorem 2.2. Let's recall the standard fact (which we have mentioned before) that under \mathbb{P}'_n , $X_k \xrightarrow{d} \text{Pois} \left(\frac{(a+b)^k}{k2^{k+1}} \right)$. With this and Theorem 3.1 in mind,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} X_k, \text{Var}_{\mathbb{P}} X_k &\rightarrow \frac{(a+b)^k + (a-b)^k}{k2^{k+1}} \\ \mathbb{E}_{\mathbb{P}'} X_k, \text{Var}_{\mathbb{P}'} X_k &\rightarrow \frac{(a+b)^k}{k2^{k+1}}. \end{aligned}$$

Set $k = k(n) = \log^{1/4} n$ (although any sufficiently slowly increasing function of n would do). Choose ρ such that $\frac{a-b}{2} > \rho > \sqrt{\frac{a+b}{2}}$. Then $\text{Var}_{\mathbb{P}} X_k$ and $\text{Var}_{\mathbb{P}'} X_k$ are both $o(\rho^{2k})$ as $k \rightarrow \infty$. By Chebyshev's inequality, $X_k \leq \mathbb{E}_{\mathbb{P}'} X_k + \rho^k$ \mathbb{P}' -a.a.s. and $X_k \geq \mathbb{E}_{\mathbb{P}} X_k - \rho^k$ \mathbb{P} -a.a.s. Since $\mathbb{E}_{\mathbb{P}} X_k - \mathbb{E}_{\mathbb{P}'} X_k = \frac{1}{2k} \left(\frac{a-b}{2} \right)^k = \omega(\rho^k)$, it follows that $\mathbb{E}_{\mathbb{P}} X_k - \rho^k \geq \mathbb{E}_{\mathbb{P}'} X_k + \rho^k$ for large enough k . And so, if we set $A_n = \{X_{k(n)} \leq \mathbb{E}_{\mathbb{P}'} X_{k(n)} + \rho^k\}$ then $\mathbb{P}'(A_n) \rightarrow 1$ and $\mathbb{P}(A_n) \rightarrow 0$. \square

Now we will prove Theorem 3.1 using the method of moments. Recall, therefore, that if $Y \sim \text{Pois}(\lambda)$ then $\mathbb{E} Y_{[m]} = \lambda^m$, where $Y_{[m]}$ denotes the falling factorial $Y(Y-1)\cdots(Y-m+1)$. It will therefore be our goal to show that $\mathbb{E}(X_k)_{[m]} \rightarrow \left(\frac{(a+b)^k + (a-b)^k}{k2^{k+1}} \right)^m$. It turns out that this follows almost entirely from the corresponding proof for the Erdős-Renyi model. The only additional work we need to do is in the case $m = 1$.

Lemma 3.2. *If $k = o(\sqrt{n})$ then*

$$\mathbb{E}_{\mathbb{P}} X_k = \binom{n}{k} \frac{(k-1)!}{2} (2n)^{-k} ((a+b)^k + (a-b)^k) \sim \frac{1}{k2^{k+1}} ((a+b)^k + (a-b)^k).$$

Proof. Let v_0, \dots, v_{k-1} be distinct vertices. Let Y be the indicator that $v_0 \dots v_{k-1}$ is a cycle in G . Then $\mathbb{E}_{\mathbb{P}} X_k = \binom{n}{k} \frac{(k-1)!}{2} \mathbb{E}_{\mathbb{P}} Y$, so let us compute

$\mathbb{E}_{\mathbb{P}} Y$. Define N to be the number of times in the cycle $v_1 \dots v_k$ that $\sigma_{v_i} \neq \sigma_{v_{i+1}}$ (with addition taken modulo k). Then

$$\mathbb{E}_{\mathbb{P}} Y = \sum_{m=0}^k \mathbb{P}(N = m) \mathbb{P}((v_1 \dots v_k) \in G | N = m) = n^{-k} \sum_{m=0}^k \mathbb{P}(N = m) a^{k-m} b^m.$$

On the other hand, we can easily compute $\mathbb{P}(N = m)$: for each $i = 0, \dots, k-2$, there is probability $\frac{1}{2}$ to have $\sigma_{v_i} = \sigma_{v_{i+1}}$, and these events are mutually independent. But whether $\sigma_{v_{k-1}} = \sigma_{v_0}$ is completely determined by the other events since there must be an even number of $i \in \{0, \dots, k-1\}$ such that $\sigma_{v_i} \neq \sigma_{v_{i+1}}$. Thus,

$$\begin{aligned} \mathbb{P}(N = m) &= \Pr\left(\text{Binom}\left(k-1, \frac{1}{2}\right) \in \{m-1, m\}\right) \\ &= 2^{-k+1} \left(\binom{k-1}{m-1} + \binom{k-1}{m} \right) = 2^{-k+1} \binom{k}{m} \end{aligned}$$

for even m , and zero otherwise. Hence,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} Y &= n^{-k} 2^{-k+1} \sum_{m \text{ even}} a^{k-m} b^m \binom{k}{m} \\ &= n^{-k} 2^{-k} \left((a+b)^k + (a-b)^k \right). \end{aligned}$$

The second part of the claim amounts to saying that $n_{[k]} \sim n^k$, which is trivial when $k = o(\sqrt{n})$. \square

Proof of Theorem 3.1. Let $\mu = \frac{1}{k2^k} \left((a+b)^k + (a-b)^k \right)$; our goal, as discussed before Lemma 3.2, is to show that $\mathbb{E}(X_k)_{[m]} \rightarrow \mu^m$. Note that $(X_k)_{[m]}$ is the number of ordered m -tuples of k -cycles in G . We will divide these m -tuples into two sets: A is the set of m -tuples for which all of the k -cycles are disjoint, while B is the set of m -tuples in which at least one pair of cycles is not disjoint.

Now, take $(C_1, \dots, C_m) \in A$. Since the C_i are disjoint, they appear independently. By the proof of Lemma 3.2, the probability that cycles C_1, \dots, C_m are all present is

$$n^{-km} 2^{-km} \left((a+b)^k + (a-b)^k \right)^m.$$

Since there are $\binom{n}{km} \frac{(km)!}{k^m}$ elements of A , it follows that the expected number of vertex-disjoint m -tuples of k -cycles is

$$\binom{n}{km} \frac{(km)!}{k^m} n^{-km} 2^{-km} \left((a+b)^k + (a-b)^k \right)^m \sim \mu^m.$$

It remains to show, therefore, that the expected number of non-vertex-disjoint m -tuples converges to zero. Let Y be the number of non-vertex-disjoint m -tuples,

$$Y = \sum_{(C_1, \dots, C_m) \in B} \prod_{i=1}^m 1_{\{C_i \subset G\}}.$$

Then the distribution of Y under \mathbb{P} is stochastically dominated by the distribution of Y under the Erdős-Renyi model $\mathcal{G}(n, \frac{\max\{a,b\}}{n})$. It's well-known (see, eg. [3], Chapter 4) that as long as $k = O(\log^{1/4} n)$, $\mathbb{E}Y \rightarrow 0$ under $\mathcal{G}(n, \frac{c}{n})$ for any c ; hence $\mathbb{E}Y \rightarrow 0$ under \mathbb{P} also. \square

4 Finding a density

In this section, we will prove the second part of Theorem 2.2. The general direction of this proof was already described in the introduction, but let's begin here with a slightly more detailed overview. Recall that \mathbb{P}'_n denotes the Erdős-Renyi model $\mathcal{G}(n, \frac{a+b}{2n})$. The first thing we will do is to extend \mathbb{P}'_n to be a distribution on labelled graphs. In order to do this, we only need to describe the conditional distribution of the label given the graph. We will take

$$\mathbb{P}'_n(\sigma|G) = \frac{\mathbb{P}_n(G|\sigma)}{Z_n(G)},$$

where $Z_n(G)$ is the normalization constant for which this is a probability. Now, our goal is to show that $\frac{\mathbb{P}_n(G, \sigma)}{\mathbb{P}'_n(G, \sigma)}$ is well-behaved; with our definition of $\mathbb{P}'_n(\sigma|G)$, we have

$$\frac{\mathbb{P}_n(G, \sigma)}{\mathbb{P}'_n(G, \sigma)} = \frac{\mathbb{P}_n(\sigma)Z_n(G)}{\mathbb{P}'_n(G)} = 2^{-n} \frac{Z_n(G)}{\mathbb{P}'_n(G)}.$$

Thus, the second part of Theorem 2.2 reduces to the study of the partition function $Z_n(G)$. To do this, we will use the small subgraph conditioning method. This method was developed by Robinson and Wormald [23, 24] in order to prove that most d -regular graphs are Hamiltonian, but it has since been applied in many different settings (see the survey [27] for a more detailed discussion). Essentially, the method is useful for studying a sequence $Y_n(G_n)$ of random variables which are not concentrated around their means, but which become concentrated when we condition on the number of short cycles that G_n has. Fortunately for us, this method has been developed into an easily applicable tool, the application of which only requires the calculation of some joint moments. The formulation below comes from [27], Theorem 4.1.

Theorem 4.1. Fix two sequences of probability distributions \mathbb{P}'_n and \mathbb{P}_n on a common sequence of discrete measure spaces, and let $Y_n = \frac{\mathbb{P}_n}{\mathbb{P}'_n}$ be the density of \mathbb{P}_n with respect to \mathbb{P}'_n . Let $\lambda_k > 0$ and $\delta_k \geq -1$ be real numbers. For each n , suppose that there are random variables $X_k = X_k(n) \in \mathbb{N}$ for $k \geq 3$ such that

(a) For each fixed $m \geq 1$, $\{X_k(n)\}_{k=3}^m$ converge jointly under \mathbb{P}'_n to independent Poisson variables with means λ_k ;

(b) For every $j_1, \dots, j_m \in \mathbb{N}$,

$$\frac{\mathbb{E}_{\mathbb{P}'_n}(Y_n[X_3(n)]_{j_1} \cdots [X_m(n)]_{j_m})}{\mathbb{E}_{\mathbb{P}'_n} Y_n} \rightarrow \prod_{k=3}^m (\lambda_k (1 + \delta_k))^{j_k};$$

(c)

$$\sum_{k \geq 3} \lambda_k \delta_k^2 < \infty;$$

(d)

$$\frac{\mathbb{E}_{\mathbb{P}'_n} Y_n^2}{(\mathbb{E}_{\mathbb{P}'_n} Y_n)^2} \rightarrow \exp \left(\sum_{k \geq 3} \lambda_k \delta_k^2 \right).$$

Then \mathbb{P}'_n and \mathbb{P}_n are contiguous.

In our application of Theorem 4.1 the discussion at the beginning of this section implies that $Y_n = Y_n(G) = 2^{-n} \frac{Z_n(G)}{\mathbb{P}'_n(G)}$. We will take $X_k(n)$ to be the number of k -cycles in G_n . Thus, condition (a) in Theorem 4.1 is already well-known, with $\lambda_k = \frac{1}{2k} \left(\frac{a+b}{2} \right)^k$. This leaves us with three conditions to check. We will start with (d), but before we do so, let us fix some notation.

Let σ and τ be two labellings in $\{\pm\}^n$. Take $S = S(\sigma) = \{u : \sigma_u = +\}$ and $T = T(\tau) = \{u : \tau_u = +\}$. It is common practice to write $|S|$ for the cardinality of S , but we will often omit the $|\cdot|$ symbol, especially where S appears in an exponent. We will also omit the subscript n in \mathbb{P}_n and \mathbb{P}'_n , and when we write $\prod_{(u,v)}$, we mean that u and v range over all unordered pairs of distinct vertices $u, v \in G$. Let t (for “threshold”) be defined by $t = \frac{(a-b)^2}{2(a+b)}$.

For the rest of this section, $G \sim \mathbb{P}'$. Therefore we will drop the \mathbb{P}' from $\mathbb{E}_{\mathbb{P}'}$ and just write \mathbb{E} .

4.1 The first two moments of Y_n

Since $Y_n = \frac{\mathbb{P}(G, \sigma)}{\mathbb{P}'(G, \sigma)}$, $\mathbb{E}Y_n = 1$ trivially. Let's do a short computation to double-check it, though, because it will be useful later. Define

$$W_{uv} = W_{uv}(G, \sigma) = \begin{cases} \frac{2a}{a+b} & \text{if } (u, v) \in E \cap S \\ \frac{2b}{a+b} & \text{if } (u, v) \in E \cap S^c \\ \frac{n-a}{n-(a+b)/2} & \text{if } (u, v) \in E^c \cap S \\ \frac{n-b}{n-(a+b)/2} & \text{if } (u, v) \in E^c \cap S^c \end{cases}$$

and define V_{uv} by the same formula, but with σ and S replaced by τ and T . Then

$$Y_n = 2^{-n} \sum_{\sigma \in \{\pm\}^n} \prod_{(u,v)} W_{uv}$$

and

$$Y_n^2 = 2^{-2n} \sum_{\sigma, \tau \in \{\pm\}^n} \prod_{(u,v)} W_{uv} V_{uv}.$$

Since $\{W_{uv}\}_{(u,v)}$ are independent given σ , it follows that

$$\mathbb{E}Y_n = 2^{-n} \sum_{\sigma \in \{\pm\}^n} \prod_{(u,v)} \mathbb{E}W_{uv} \quad (1)$$

and

$$\mathbb{E}Y_n^2 = 2^{-2n} \sum_{\sigma, \tau \in \{\pm\}^n} \prod_{(u,v)} \mathbb{E}W_{uv} V_{uv}. \quad (2)$$

Thus, to compute $\mathbb{E}Y_n$, we should compute $\mathbb{E}W_{uv}$, while computing $\mathbb{E}Y_n^2$ involves computing $\mathbb{E}W_{uv} V_{uv}$.

Lemma 4.2. *For any fixed σ ,*

$$\mathbb{E}W_{uv}(G, \sigma) = 1.$$

Proof. Suppose $(u, v) \in S$. Then $\mathbb{P}'((u, v) \in E) = \frac{a+b}{2n}$, so

$$\mathbb{E}W_{uv} = \frac{2a}{a+b} \cdot \frac{a+b}{2n} + \frac{n-a}{n-(a+b)/2} \cdot \left(1 - \frac{a+b}{2n}\right) = \frac{a}{n} + 1 - \frac{a}{n} = 1.$$

The case for $(u, v) \in S^c$ is similar. \square

Nonwithstanding the fact that computing $\mathbb{E}Y_n$ is trivial anyway, Lemma 4.2 and (1) together imply that $\mathbb{E}Y_n = 1$. Let us now move on to the second moment.

Lemma 4.3. *If $(u, v) \in S \cap T$ or $(u, v) \in S^c \cap T^c$ then*

$$\mathbb{E}W_{uv}V_{uv} = 1 + \frac{1}{n} \cdot \frac{(a-b)^2}{2(a+b)} + \frac{(a-b)^2}{4n^2} + O(n^{-3}).$$

Otherwise,

$$\mathbb{E}W_{uv}V_{uv} = 1 - \frac{1}{n} \cdot \frac{(a-b)^2}{2(a+b)} - \frac{(a-b)^2}{4n^2} + O(n^{-3}).$$

Proof. Suppose $(u, v) \in S \cap T$. Then

$$\begin{aligned} \mathbb{E}W_{uv}V_{uv} &= \left(\frac{2a}{a+b}\right)^2 \cdot \frac{a+b}{2n} + \left(\frac{n-a}{n-(a+b)/2}\right)^2 \cdot \left(1 - \frac{a+b}{2n}\right) \\ &= \frac{2a^2}{n(a+b)} + \frac{\left(1 - \frac{a}{n}\right)^2}{1 - \frac{a+b}{2n}} \\ &= \frac{2a^2}{n(a+b)} + \left(1 - \frac{a}{n}\right)^2 \left(1 + \frac{a+b}{2n} + \frac{(a+b)^2}{4n^2} + O(n^{-3})\right) \\ &= 1 + \frac{1}{n} \cdot \frac{(a-b)^2}{2(a+b)} + \frac{(a-b)^2}{4n^2} + O(n^{-3}). \end{aligned}$$

The computation for $(u, v) \in S^c \cap T^c$ is analogous.

Now assume $(u, v) \in S \cap T^c$. By a very similar computation,

$$\begin{aligned} \mathbb{E}W_{uv}V_{uv} &= \frac{4ab}{(a+b)^2} \cdot \frac{a+b}{2n} + \frac{\left(1 - \frac{a}{n}\right)\left(1 - \frac{b}{n}\right)}{\left(1 - \frac{a+b}{2n}\right)^2} \left(1 - \frac{a+b}{2n}\right) \\ &= 1 - \frac{1}{n} \cdot \frac{(a-b)^2}{2(a+b)} - \frac{(a-b)^2}{4n^2} + O(n^{-3}). \end{aligned}$$

The computation for $(u, v) \in S^c \cap T$ is analogous. \square

Given what we said just before Lemma 4.2, we can now compute $\mathbb{E}Y_n^2$ just by looking at the sizes of $S \cap T$, $S^c \cap T$ and so on. To make this easier, we introduce another parameter, $\rho = \rho(\sigma, \tau) = \frac{1}{n} \sum_i \sigma_i \tau_i$. Then some elementary counting gives

$$\begin{aligned} |S \cap T| + |S^c \cap T^c| &= (1 + \rho^2) \frac{n^2}{4} - \frac{n}{2} \\ |S \cap T^c| + |S^c \cap T| &= (1 - \rho^2) \frac{n^2}{4}. \end{aligned}$$

Lemma 4.4.

$$\mathbb{E}Y_n^2 = (1 + o(1)) \frac{e^{-t/2-t^2/4}}{\sqrt{1-t}}.$$

Before we proceed to the proof, recall (or check, by writing out the Taylor series of the logarithm) that

$$\left(1 + \frac{x}{n}\right)^{n^2} = (1 + o(1)) e^{nx - \frac{1}{2}x^2}$$

as $n \rightarrow \infty$.

Proof. Define $\gamma_n = \frac{t}{n} + \frac{(a-b)^2}{4n^2}$; note that

$$\begin{aligned} (1 + \gamma_n)^{n^2} &= (1 + o(1)) \exp\left(\frac{(a-b)^2}{4} + tn - \frac{t^2}{2}\right) \\ (1 - \gamma_n)^{n^2} &= (1 + o(1)) \exp\left(-\frac{(a-b)^2}{4} - tn - \frac{t^2}{2}\right) \\ (1 + \gamma_n)^n &= (1 + o(1)) \exp(t). \end{aligned}$$

Then, by Lemma 4.3,

$$\begin{aligned} 2^{2n} \mathbb{E}Y_n^2 &= \sum_{\sigma, \tau} \prod_{(u,v)} \mathbb{E}W_{uv} V_{uv} \\ &= \sum_{\sigma, \tau} (1 + \gamma_n + O(n^{-3}))^{S \cap T + S^c \cap T^c} (1 - \gamma_n + O(n^{-3}))^{S^c \cap T + S \cap T^c} \\ &= (1 + o(1)) e^{-t/2} \sum_{\sigma, \tau} (1 + \gamma_n)^{(1+\rho^2)n^2/4} (1 - \gamma_n)^{(1-\rho^2)n^2/4} \\ &= (1 + o(1)) e^{-t/2-t^2/4} \sum_{\sigma, \tau} \exp\left(\frac{\rho^2}{2} \left(\frac{(a-b)^2}{4} + tn\right)\right). \end{aligned}$$

Computing the last term would be easy if $\rho\sqrt{n}$ were normally distributed. Instead, it is binomially distributed, which – unsurprisingly – is just as good. To show it, though, will require a slight digression.

Lemma 4.5. *If $\xi_i \in \{\pm 1\}$ are taken uniformly and independently at random and $Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i$ then*

$$\mathbb{E} \exp(s Z_n^2 / 2) \rightarrow \frac{1}{\sqrt{1-s}}$$

whenever $s < 1$.

Proof. Since $z \mapsto \exp(sz^2/2)$ is a continuous function, the central limit theorem implies that $\exp(sZ_n^2/2) \xrightarrow{d} \exp(sZ^2/2)$, where $Z \sim \mathcal{N}(0,1)$. Now, $\mathbb{E} \exp(sZ^2/2) = \frac{1}{\sqrt{1-s}}$ and so the proof is complete if we can show that the sequence $\exp(sZ_n^2/2)$ is uniformly integrable. But this follows from Hoeffding's inequality:

$$\Pr(\exp(sZ_n^2/2) \geq M) = \Pr\left(|Z_n| \geq \sqrt{\frac{2 \log M}{s}}\right) \leq M^{-1/s},$$

which is integrable near ∞ (uniformly in n) whenever $s < 1$. \square

To finish the proof of Lemma 4.4, take Z_n as in Lemma 4.5 and note that

$$2^{-2n} \sum_{\sigma, \tau} \exp\left(\frac{\rho^2}{2} \left(\frac{(a-b)^2}{4} + tn\right)\right) = \mathbb{E} \exp\left(\frac{t(1+o(1))}{2} Z_n^2\right) \rightarrow \frac{1}{\sqrt{1-t}}. \quad \square$$

4.2 Dependence on the number of short cycles

Our next task is to check condition (b) in Theorem 4.1. Note, therefore, that $[X_3]_{j_3} \cdots [X_m]_{j_m}$ is the number of ways to have an ordered tuple containing j_3 3-cycles of G , j_4 4-cycles of G , and so on. Therefore, if we can compute $\mathbb{E} Y_n 1_H$ where 1_H indicates that any particular union of cycles occurs in G_n , then we can compute $\mathbb{E} Y_n [X_3]_{m_3} \cdots [X_m]_{j_m}$. Computing $\mathbb{E} Y_n 1_H$ is the main task of this section; we will do it in three steps. First, we will get a general formula for $\mathbb{E} Y_n 1_H$ in terms of H . We will apply this general formula in the case that H is a single cycle and get a much simpler formula back. Finally, we will extend this to the case when H is a union of vertex-disjoint cycles.

As promised, we begin the program with a general formula for $\mathbb{E} 1_H Y_n$. Let H be a graph on some subset of $[n]$, with $|V(H)| = m$. With some slight abuse of notation, We write 1_H for the random variable that is 1 when $H \subset G$, and $\mathbb{P}'(H)$ for the probability that $H \subset G$.

Lemma 4.6.

$$\mathbb{E} 1_H Y_n = 2^{-m} \mathbb{P}'(H) \sum_{\sigma \in \{\pm 1\}^m} \prod_{(u,v) \in E(H)} w_{uv}(\sigma),$$

where

$$w_{uv}(\sigma) = \begin{cases} \frac{2a}{a+b} & \text{if } (u,v) \in S(\sigma) \\ \frac{2b}{a+b} & \text{otherwise.} \end{cases}$$

Proof. We break up $\sigma \in \{\pm 1\}^n$ into $(\sigma_1, \sigma_2) \in \{\pm 1\}^{V(H)} \times \{\pm 1\}^{V(G) \setminus V(H)}$ and sum over the two parts separately. Note that if $(u, v) \in E(H)$ then $W_{uv}(G, \sigma)$ depends on σ only through σ_1 . Let $D(H) = E(G) \setminus E(H)$, so that $(u, v) \in D(H)$ implies that W_{uv} and 1_H are independent. Then

$$\begin{aligned} \mathbb{E} 1_H Y_n &= 2^{-n} \sum_{\sigma_1} \sum_{\sigma_2} \mathbb{E} 1_H \prod_{(u,v)} W_{uv}(G, \sigma) \\ &= 2^{-n} \sum_{\sigma_1} \left(\left(\mathbb{E} 1_H \prod_{(u,v) \in E(H)} W_{uv} \right) \sum_{\sigma_2} \prod_{(u,v) \in D(H)} \mathbb{E} W_{uv} \right) \\ &= 2^{-m} \sum_{\sigma_1} \left(\mathbb{E} 1_H \prod_{(u,v) \in E(H)} W_{uv} \right), \end{aligned}$$

because if $(u, v) \in D(H)$ then, for every σ , Lemma 4.2 says that $\mathbb{E} W_{uv}(G, \sigma) = 1$. To complete the proof, note that if $(u, v) \in E(H)$ then for any σ , $W_{uv}(G, \sigma) \equiv w_{uv}(\sigma)$ on the event $H \subseteq G$. \square

The next step is to compute the right hand side of Lemma 4.6 in the case that H is a cycle. This computation is very similar to the one in Lemma 3.2, when we computed the expected number of k -cycles in $\mathcal{G}(n, \frac{a}{n}, \frac{b}{n})$. Essentially, we want to compute the expected “weight” of a cycle, where the weight of each edge depends only on whether its endpoints have the same label or not.

Lemma 4.7. *If H is a k -cycle then*

$$\sum_{\sigma \in \{\pm 1\}^H} \prod_{(u,v) \in E(H)} w_{uv}(\sigma) = 2^k \left(1 + \left(\frac{a-b}{a+b} \right)^k \right).$$

Proof. Let e_1, \dots, e_k be the edges of H . Provided that we renormalize, we can replace the sum over σ by an expectation, where σ is taken uniformly in $\{\pm 1\}^H$. Now, let N be the number of edges of H whose endpoints have different labels. As discussed in the proof of Lemma 3.2, $\Pr(N = j) = 2^{-k+1} \binom{k}{j}$ for even j , and zero otherwise. Then

$$\begin{aligned} \mathbb{E}_\sigma \prod_{(u,v) \in E(H)} w_{uv}(\sigma) &= \mathbb{E}_\sigma \left(\frac{2a}{a+b} \right)^{k-N} \left(\frac{2b}{a+b} \right)^N \\ &= \frac{2}{(a+b)^k} \sum_{j \text{ even}} \binom{k}{j} a^{k-j} b^j \\ &= 1 + \left(\frac{a-b}{a+b} \right)^k. \end{aligned} \quad \square$$

Extending this calculation to vertex-disjoint unions of cycles is quite easy: suppose H is the union of cycles H_i . Since $w_{uv}(\sigma)$ only depends on σ_u and σ_v , we can just split up the sum over $\sigma \in \{\pm\}^H$ into a product of sums, where each sum ranges over $\{\pm\}^{H_i}$. Then applying Lemma 4.7 to each H_i yields a formula for H .

Lemma 4.8. *Define*

$$\delta_k = \left(\frac{a-b}{a+b} \right)^k.$$

If $H = \cup_i H_i$ is a vertex-disjoint union of graphs and each H_i is a k_i -cycle, then

$$\sum_{\sigma \in \{\pm 1\}^H} \prod_{(u,v) \in E(H)} w_{uv}(H, \sigma) = 2^{|H|} \prod_i (1 + \delta_{k_i}).$$

We need one last ingredient, which we hinted at earlier, before we can show condition (b) of Theorem 4.1. We only know how to exactly compute $\mathbb{E}Y_n 1_H$ when H is a disjoint union of cycles. Now, most tuples of cycles are disjoint, but in order to dismiss the contributions from the non-disjoint unions, we need some bound on $\mathbb{E}Y_n 1_H$ that holds for all H :

Lemma 4.9. *For any H ,*

$$\sum_{\sigma \in \{\pm 1\}^H} \prod_{(u,v) \in E(H)} w_{uv}(\sigma) \leq 2^{|H| + |E(H)|}.$$

Proof.

$$w_{uv}(\sigma) \leq \frac{2 \max\{a, b\}}{a+b} \leq 2$$

for any i, j, H and σ . □

Finally, we are ready to put these ingredients together and prove condition (b) of Theorem 4.1. For the rest of the section, take $\delta_k = \left(\frac{a-b}{a+b} \right)^k$ as it was in Lemma 4.8. Also, recall that $\lambda_k = \frac{1}{2k} \left(\frac{a+b}{2} \right)^k$ is the limit of $\mathbb{E}X_k$ as $n \rightarrow \infty$.

Lemma 4.10. *Let X_k be the number of k -cycles in G . For any $j_3, \dots, j_m \in \mathbb{N}$,*

$$\mathbb{E}Y_n \prod_{k=3}^m [X_k]_{j_k} \rightarrow \prod_{k=3}^m (\lambda_k (1 + \delta_k))^{j_k}.$$

Proof. Set $M = \sum_k km_k$. First of all,

$$[X_k]_j = \sum_{H_1, \dots, H_j} \prod_i 1_{H_i}$$

where the sum ranges over all j -tuples of distinct k -cycles, and 1_H indicates the event that the subgraph H appears in G . Thus,

$$\prod_{k=3}^m [X_k]_{j_k} = \sum_{(H_{ki})} \prod_{k=3}^m \prod_{i=1}^{j_k} 1_{H_{ki}} = \sum_{(H_{ki})} 1_{\{\cup H_{ki}\}},$$

where the sum ranges over all M -tuples of cycles $(H_{ki})_{k \leq m, i \leq j_k}$ for which each H_{ki} is an k -cycle, and every cycle is distinct. Let \mathcal{H} be the set of such tuples; let $A \subset \mathcal{H}$ be the set of such tuples for which the cycles are vertex-disjoint, and let $B = \mathcal{H} \setminus A$. Thus, if $H = \cup H_{ki}$ for $(H_{ki}) \in A$, then

$$\mathbb{E}Y_n 1_H = \prod_k (1 + \delta_k)^{j_k} \mathbb{P}'(H)$$

by Lemmas 4.6 and 4.8. Note also that standard counting arguments (see, for example, [3], Chapter 4) imply that $|A| \mathbb{P}'(H) \rightarrow \prod_k \lambda_k^{j_k}$.

On the other hand, if $(H_{ki}) \in B$ then $H := \cup_{ki} H_{ki}$ has at most $M - 1$ vertices, M edges, and its number of edges is strictly larger than its number of vertices. Thus, $\mathbb{P}'(H) \binom{n}{|H|} \rightarrow 0$, so Lemmas 4.6 and 4.9 imply that

$$\sum_{H' \sim H} \mathbb{E}Y_n 1_{H'} \leq \mathbb{P}'(H) |H|! \binom{n}{|H|} 2^M \rightarrow 0,$$

where the sum ranges over all ways to make an isomorphic copy of H on n vertices. Since there are only a bounded number of isomorphism classes in

$$\left\{ \bigcup_{ki} H_{ki} : (H_{ki}) \in B \right\},$$

it follows that $\sum_H \mathbb{E}Y_n 1_H \rightarrow 0$, where the sum ranges over all unions of non-disjoint tuples in \mathcal{H} . Thus,

$$\begin{aligned} \mathbb{E}Y_n \prod_{k=3}^m [X_k]_{j_k} &= \mathbb{E}Y_n \left(\sum_{(H_{ki}) \in A} 1_{\cup H_{ki}} + \sum_{(H_{ki}) \notin A} 1_{\cup H_{ki}} \right) \\ &= |A| \mathbb{P}'(H) \prod_k (1 + \delta_k)^{j_k} + o(1) \\ &\rightarrow \prod_k (\lambda_k (1 + \delta_k))^{j_k}. \end{aligned} \quad \square$$

To complete the proof of Theorem 2.2, note that $\delta_k^2 \lambda_k = \frac{t^k}{2k}$. Thus, $\sum_{k \geq 3} \delta_k^2 \lambda_k = \frac{1}{2}(\log(1-t) - t - t^2/2)$. When $t < 1$, this (with Lemma 4.4) proves conditions (c) and (d) of Theorem 4.1. Since condition (a) is classical and condition (b) is given by Lemma 4.10, the conclusion of Theorem 4.1 implies the second part of Theorem 2.2.

5 Non-reconstruction

The goal of this section is to prove Theorem 2.1. As we said in the introduction, the proof of Theorem 2.1 uses a connection between $\mathcal{G}(n, \frac{a}{n}, \frac{b}{n})$ and Markov processes on trees. Before we go any further, therefore, we should define a Markov process on a tree and state the result that we will use.

Let T be an infinite rooted tree with root ρ . Given a number $0 \leq \epsilon < 1$, we will define a random labelling $\tau \in \{\pm\}^T$. First, we draw τ_ρ uniformly in $\{\pm\}$. Then, conditionally independently given τ_ρ , we take every child u of ρ and set $\tau_u = \tau_\rho$ with probability $1 - \epsilon$ and $\tau_u = -\tau_\rho$ otherwise. We can continue this construction recursively to obtain a labelling τ for which every vertex, independently, has probability $1 - \epsilon$ of having the same label as its parent.

Back in 1966, Kesten and Stigum [19] asked (although they used somewhat different terminology) whether the label of ρ could be deduced from the labels of vertices at level R of the tree (where R is very large). There are many equivalent ways of stating the question. The interested reader should see the survey [22], because we will only mention two of them.

Let $T_R = \{u \in T : d(u, \rho) \leq R\}$ and define $\partial T_R = \{u \in T : d(u, \rho) = R\}$. We will write τ_{T_R} for the configuration τ restricted to T_R .

Theorem 5.1. *Suppose T is a Galton-Watson tree where the offspring distribution has mean $d > 1$. Then*

$$\lim_{R \rightarrow \infty} \Pr(\tau_\rho = + | \tau_{\partial T_R}) = \frac{1}{2} \text{ a.s.}$$

if, and only if $d(1 - 2\epsilon)^2 \leq 1$.

In particular, if $d(1 - 2\epsilon)^2 \leq 1$ then $\tau_{\partial T_R}$ contains no information about τ_ρ . Theorem 5.1 was established by several authors over the course of more than 30 years. The non-reconstruction regime (ie. the case $d(1 - 2\epsilon)^2 \leq 1$) is the harder one, and that part of Theorem 5.1 was first proved for d -ary trees in [2], and for Galton-Watson trees in [12]. This latter work actually proves the result for more general trees in terms of their branching number.

We will be interested in trees T whose offspring distribution is $\text{Pois}(\frac{a+b}{2})$ and we will take $1 - \epsilon = \frac{a}{a+b}$. Some simple arithmetic applied to Theorem 5.1 then shows that reconstruction of the root's label is impossible whenever $(a-b)^2 \leq 2(a+b)$. Not coincidentally, this is the same threshold that appears in Theorem 2.1.

5.1 A neighborhood of G looks like T

The first step in applying Theorem 5.1 to our problem is to observe that a neighborhood of $(G, \sigma) \sim \mathcal{G}(n, \frac{a}{n}, \frac{b}{n})$ looks like (T, τ) . Indeed, fix $\rho \in G$ and let G_R be the induced subgraph on $\{u \in G : d(u, \rho) \leq R\}$.

Proposition 5.2. *Let $R = R(n) = \lfloor \frac{1}{10 \log(2(a+b))} \log n \rfloor$. There exists a coupling between (G, σ) and (T, τ) such that $(G_R, \sigma_{G_R}) = (T_R, \tau_{T_R})$ a.a.s.*

For the rest of this section, we will take $R = \lfloor \frac{1}{8 \log(2(a+b))} \log n \rfloor$.

The proof of this lemma essentially follows from the fact that (T, τ) can be constructed from a sequence of independent Poisson variables, while (G_R, σ_{G_R}) can be constructed from a sequence of binomial variables, with approximately the same means.

For a vertex $v \in T$, let Y_v be the number of children of v ; let Y_v^- be the number of children whose label is τ_v and let $Y_v^+ = Y_v - Y_v^-$. By Poisson thinning, $Y_v^- \sim \text{Pois}(a/2)$, $Y_v^+ \sim \text{Pois}(b/2)$ and they are independent. Note that (T, τ) can be entirely reconstructed from the label of the root and the two sequences (Y_i^-) , (Y_i^+) .

We can almost do the same thing for G_R , but it is a little more complicated. We will write $V = V(G)$ and $V_R = V(G) \setminus V(G_R)$. For every subset $W \subset V$, denote by W^+ and W^- the subsets of W that have the corresponding label. For example, $V_R^+ = \{v \in V_R : \sigma_v = +\}$. For a vertex $v \in \partial G_R$, let X_v be the number of neighbors that v has in V_r ; then let X_v^- be the number of those neighbors whose label is σ_v and set $X_v^+ = X_v - X_v^-$. Then $X_v^- \sim \text{Binom}(|V_r^{\sigma_v}|, a)$, $X_v^+ \sim \text{Binom}(|V_r^{-\sigma_v}|, b)$ and they are independent. Note, however, that they do not contain enough information to reconstruct G_R : it's possible to have $u, v \in \partial G_r$ which share a child in V_r , but this cannot be determined from X_u and X_v . Fortunately, such events are very rare and so we can exclude them. In fact, this process of carefully excluding bad events is all that needs to be done to prove Proposition 5.2.

In order that we can exclude their complements, let us give names to all of our good events. For any r , let A_r be the event that no vertex in V_r has more than one neighbor in G_r . Let B_r be the event that there are no edges within ∂G_r . Clearly, if A_r and B_r hold for all $r = 1, \dots, R$ then G_R

is a tree. In fact, it's easy to see that A_r and B_r are the only events that prevent $\{X_v^-, X_v^\#\}_{v \in G}$ from determining (G_R, σ_{G_R}) .

Lemma 5.3. *If*

1. $(T_{r-1}, \tau_{T_{r-1}}) = (G_{r-1}, \sigma_{G_{r-1}})$;
2. $X_u^- = Y_u^-$ and $X_u^\# = Y_u^\#$ for every $u \in \partial G_{r-1}$; and
3. A_r and B_r hold

then $(T_r, \tau_{T_r}) = (G_r, \sigma_{G_r})$.

Proof. The proof is essentially obvious from the construction of X_u and Y_u , but we will be pedantic about it anyway. The statement $(T_{r-1}, \tau_{T_{r-1}}) = (G_{r-1}, \sigma_{G_{r-1}})$ means that there is some graph homomorphism $\phi : G_{r-1} \rightarrow T_{r-1}$ such that $\sigma_u = \tau_{\phi(u)}$. If $u \in \partial G_{r-1}$ and $X_u^- = Y_{\phi(u)}^-$ and $X_u^\# = Y_{\phi(u)}^\#$ then we can extend ϕ to $G_{r-1} \cup \mathcal{N}(u)$ while preserving the fact that $\sigma_v = \tau_{\phi(v)}$ for all v . On the event A_r , this extension can be made simultaneously for all $u \in \partial G_{r-1}$, while the event B_r ensures that this extension remains a homomorphism. Thus, we have constructed a label-preserving homomorphism from (G_r, σ_{G_r}) to (T_r, τ_{T_r}) , which is the same as saying that these two labelled graphs are equal.

From now on, we will not mention homomorphisms; we will just identify u with $\phi(u)$. \square

In order to complete our coupling, we need to identify one more kind of good event. Let C_r be the event

$$C_r = \{|\partial G_s| \leq 2^s(a+b)^s \log n \text{ for all } s \leq r+1\}.$$

The events C_r are useful because they guarantee that V_r is large enough for the desired binomial-Poisson approximation to hold. The utility of C_r is demonstrated by the next two lemmas.

Lemma 5.4.

$$\mathbb{P}(C_r | C_{r-1}, \sigma) \geq 1 - n^{-\log(4/e)}.$$

Moreover, $|G_r| = O(n^{1/8})$ on C_{r-1} .

Lemma 5.5. *For any r ,*

$$\mathbb{P}(A_r | C_{r-1}, \sigma) \geq 1 - O(n^{-3/4})$$

$$\mathbb{P}(B_r | C_{r-1}, \sigma) \geq 1 - O(n^{-3/4}).$$

Proof of Lemma 5.4. First of all, X_v is stochastically dominated by $\text{Binom}(n, \frac{a+b}{n})$ for any v . On C_{r-1} , $|\partial G_r| \leq 2^r (a+b)^r \log n$ and so $|\partial G_{r+1}|$ is stochastically dominated by

$$Z \sim \text{Binom}\left(2^r (a+b)^r n \log n, \frac{a+b}{n}\right).$$

Thus,

$$\mathbb{P}(\neg C_r | C_{r-1}, \sigma) \leq \mathbb{P}(Z \geq 2\mathbb{E}Z) \leq \left(\frac{e}{4}\right)^{\mathbb{E}Z}$$

by a multiplicative version of Chernoff's inequality. But

$$\mathbb{E}Z = 2^r (a+b)^{r+1} \log n \geq \log n,$$

which proves the first part of the lemma.

For the second part, on C_{r-1}

$$|G_r| = \sum_{r=1}^R |\partial G_r| \leq \sum_{r=1}^R 2^r (a+b)^r \log n \leq (2(a+b))^{R+1} \log n = O(n^{1/8}). \quad \square$$

Proof of Lemma 5.5. For the first claim, fix $u, v \in \partial G_r$. For any $w \in V_r$, the probability that (u, w) and (v, w) both appear is $O(n^{-2})$. Now, $|V_r| \leq n$ and Lemma 5.4 implies that $|\partial G_r|^2 = O(n^{1/4})$. Hence the result follows from a union bound over all triples u, v, w .

For the second part, the probability of having an edge between any particular $u, v \in \partial G_r$ is $O(n^{-1})$. Lemma 5.4 implies that $|\partial G_r|^2 = O(n^{1/4})$ and so the result follows from a union bound over all pairs u, v . \square

The final ingredient we need is a bound on the total variation distance between binomial and Poisson random variables.

Lemma 5.6. *If m and n are positive integers then*

$$\left\| \text{Binom}\left(m, \frac{c}{n}\right) - \text{Pois}(c) \right\|_{TV} = O\left(\frac{\max\{1, |m-n|\}}{n}\right).$$

Proof. A classical result of Hodges and Le Cam [15] shows that

$$\left\| \text{Binom}\left(m, \frac{c}{n}\right) - \text{Pois}\left(\frac{mc}{n}\right) \right\|_{TV} \leq \frac{c^2 m}{n^2} = O(n^{-1}).$$

With the triangle inequality in mind, we need only show that $\text{Pois}(cm/n)$ is close to $\text{Pois}(c)$. This follows from a direct computation: if $\lambda < \mu$ then $\|\text{Pois}(\lambda) - \text{Pois}(\mu)\|_{TV}$ is just

$$\sum_{k \geq 0} \frac{|e^{-\mu} \mu^k - e^{-\lambda} \lambda^k|}{k!} \leq |e^{-\mu} - e^{-\lambda}| \sum_{k \geq 0} \frac{\mu^k}{k!} + e^{-\lambda} \sum_{k \geq 0} \frac{|\mu^k - \lambda^k|}{k!}.$$

Now the first term is $e^{\mu-\lambda} - 1$ and we can bound $\mu^k - \lambda^k \leq k(\mu - \lambda)\mu^{k-1}$ by the mean value theorem. Thus,

$$\| \text{Pois}(\lambda) - \text{Pois}(\mu) \|_{TV} \leq e^{\mu-\lambda} - 1 + e^{\mu-\lambda}(\mu - \lambda) = O(\mu - \lambda).$$

The claim follows from setting $\mu = c$ and $\lambda = \frac{cm}{n}$. \square

Finally, we are ready to prove Proposition 5.2.

Proof of Proposition 5.2. Let $\tilde{\Omega}$ be the event that $\left| |V^+| - |V^-| \right| \leq n^{3/4}$. Clearly, $\mathbb{P}(\tilde{\Omega}) \rightarrow 1$ exponentially fast.

Fix r and suppose that C_{r-1} and $\tilde{\Omega}$ hold, and that $(T_r, \tau_r) = (G_r, \sigma_r)$. Then for each $u \in \partial G_r$, X_u^- is distributed as $\text{Binom}(|V_r^{\sigma_u}|, a/n)$. Now,

$$\frac{n}{2} + n^{3/4} \geq |V^{\sigma_u}| \geq |V_r^{\sigma_u}| \geq |V^{\sigma_u}| - |G_{r-1}| \geq \frac{n}{2} - n^{3/4} - O(n^{1/8})$$

and so Lemma 5.6 implies that we can couple X_u^- with Y_u^- such that $\mathbb{P}(X_u^- \neq Y_u^-) = O(n^{-1/4})$ (and similarly for X_u^+ and Y_u^+). Since $|\partial G_{r-1}| = O(n^{1/8})$ by Lemma 5.4, we can find a coupling such that with probability at least $1 - O(n^{-1/8})$, $X_u^- = Y_u^-$ and $X_u^+ = Y_u^+$ for every $u \in \partial G_{r-1}$. Moreover, Lemmas 5.4 and 5.5 imply A_r, B_r and C_r hold simultaneously with probability at least $1 - n^{-\log(4/e)} - O(n^{-3/4})$. Putting these all together, we see that the hypothesis of Lemma 5.3 holds with probability at least $1 - O(n^{-1/8})$. Thus,

$$\mathbb{P}\left((G_{r+1}, \sigma_{r+1}) = (T_{r+1}, \tau_{r+1}), C_r \mid (G_r, \sigma_r) = (T_r, \tau_r), C_{r-1}\right) \geq 1 - O(n^{-1/8}).$$

But $\mathbb{P}(C_0) = 1$ and we can certainly couple (G_1, σ_1) with (T_1, τ_1) . Therefore, with a union bound over $r = 1, \dots, R$, we see that $(G_R, \sigma_R) = (T_R, \tau_R)$ a.a.s. \square

5.2 Non-reconstruction

We have shown that a neighborhood in G looks like a Galton-Watson tree with a Markov process on it. In this section, we will apply this fact to prove Theorem 2.1. In the statement of Theorem 2.1, we claimed that $\mathbb{E}(\sigma_\rho | G, \sigma_v) \rightarrow 0$, but this is clearly equivalent to $\text{Var}(\sigma_\rho | G, \sigma_v) \rightarrow 1$. This latter statement is the one that we will prove, because the conditional variance has a nice monotonicity property.

The idea behind the proof is to condition on the labels of ∂G_R , which can only make reconstruction easier. Then we can remove the conditioning on σ_v , because $\sigma_{\partial G_R}$ gives much more information anyway. Since Theorem 5.1

and Proposition 5.2 imply that σ_v cannot be reconstructed from $\sigma_{\partial G_R}$, we conclude that it cannot be reconstructed from σ_v either.

The first step is to prove that, once we have conditioned on $\sigma_{\partial G_R}$, we can remove the conditioning on σ_v . If $\sigma|G$ were distributed according to a Markov random field, this would be trivial because conditioning on $\sigma_{\partial G_R}$ would turn σ_v and σ_ρ independent. For our model, unfortunately, there are weak long-range interactions. However, these interactions are sufficiently weak that we can get an asymptotic independence result for separated sets as long as one of takes up most of the graph.

In what follows, we say that $X = o(a(n))$ a.a.s. if for every $\epsilon > 0$, $\Pr(|X| \geq \epsilon a(n)) \rightarrow 0$ as $n \rightarrow \infty$, and we say that $X = O(a(n))$ a.a.s. if

$$\limsup_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} \Pr(|X| \geq Ka(n)) = 0.$$

Lemma 5.7. *Let $A = A(G), B = B(G), C = C(G) \subset V$ be a (random) partition of V such that B separates A and C in G . If $|A \cup B| = o(\sqrt{n})$ for a.a.e. G*

$$\mathbb{P}(\sigma_A | \sigma_{B \cup C}, G) = (1 + o(1)) \mathbb{P}(\sigma_A | \sigma_B, G)$$

for a.a.e. G and σ .

Note that Lemma 5.7 is only true for a.a.e. σ . In particular, the lemma does not hold for σ that are very unbalanced (eg. $\sigma = +^V$).

Proof. As in the analogous proof for a Markov random field, we factorize $\mathbb{P}(G, \sigma)$ into parts depending on A, B and C . We then show that the part which measures the interaction between A and C is negligible. The rest of the proof is then quite similar to the Markov random fields case.

Define

$$\psi_{uv}(G, \sigma) = \begin{cases} \frac{a}{n} & \text{if } (u, v) \in E(G) \text{ and } \sigma_u = \sigma_v \\ \frac{b}{n} & \text{if } (u, v) \in E(G) \text{ and } \sigma_u \neq \sigma_v \\ 1 - \frac{a}{n} & \text{if } (u, v) \notin E(G) \text{ and } \sigma_u = \sigma_v \\ 1 - \frac{b}{n} & \text{if } (u, v) \notin E(G) \text{ and } \sigma_u \neq \sigma_v. \end{cases}$$

For arbitrary subsets $U_1, U_2 \subset V$, define

$$Q_{U_1, U_2} = Q_{U_1, U_2}(G, \sigma) = \prod_{u \in U_1, v \in U_2} \psi_{uv}(G, \sigma).$$

(If U_1 and U_2 overlap, the product ranges over all unordered pairs (u, v) with $u \neq v$; that is, if (u, v) is in the product then (v, u) is not.) Then

$$2^n \mathbb{P}(G, \sigma) = \mathbb{P}(G | \sigma) = Q_{A \cup B, A \cup B} Q_{B \cup C, C} Q_{A, C}. \quad (3)$$

First, we will show that $Q_{A,C}$ is essentially independent of σ . Take a deterministic sequence α_n with $\alpha_n/\sqrt{n} \rightarrow \infty$ but $\alpha_n|A| = o(n)$ a.a.s. Define $s_A(\sigma) = \sum_{v \in A} \sigma_v$ and $s_C(\sigma) = \sum_{v \in C} \sigma_v$ and let

$$\Omega = \{\tau \in \{\pm\}^V : |s_C(\tau)| \leq \alpha_n\}$$

$$\Omega_U = \Omega_U(\sigma) = \{\tau \in \{\pm\}^V : \tau_U = \sigma_U \text{ and } |s_C(\tau)| \leq \alpha_n\}.$$

By the definition of α_n , if $\tau \in \Omega$ then $|s_A(\tau)s_C(\tau)| \leq |A|\alpha_n = o(n)$ a.a.s. Thus, $\tau \in \Omega$ implies

$$\begin{aligned} Q_{A,C}(G, \tau) &= \prod_{u \in A, v \in C} \psi_{uv}(G, \tau) \\ &= \left(1 - \frac{a}{n}\right)^{(|A||C| + s_A(\tau)s_C(\tau))/2} \left(1 - \frac{b}{n}\right)^{(|A||C| - s_A(\tau)s_C(\tau))/2} \\ &= (1 + o(1)) \left(1 - \frac{a}{n}\right)^{|A||C|/2} \left(1 - \frac{b}{n}\right)^{|A||C|/2} \quad \text{a.a.s.} \end{aligned} \quad (4)$$

where we have used the fact that $u \in A, v \in C$ implies that $(u, v) \notin E(G)$, and thus ψ_{uv} is either $1 - \frac{a}{n}$ or $1 - \frac{b}{n}$. Moreover, $1 - \frac{a}{n}$ appears once for every pair $(u, v) \in A \times C$ where $\tau_u = \tau_v$. The number of such pairs is $|A_+||C_+| + |A_-||C_-|$ where $A_+ = \{u \in A : \tau_u = +\}$ (and similarly for C_+ , etc.); it's easy to check, then, that $2(|A_+||C_+| + |A_-||C_-|) = |A||C| + s_A s_C$, which explains the exponents in (4).

Note that the right hand side of (4) depends on G (through $A(G)$ and $C(G)$) but not on τ . Writing $2^{-n}K(G)$ for the right hand side of (4), (3) implies that if $\tau \in \Omega$ then

$$\mathbb{P}(G, \tau) = (1 + o(1))K(G)Q_{A \cup B, A \cup B}(G, \tau)Q_{B \cup C, C}(G, \tau) \quad (5)$$

for a.a.e. G . Moreover, $\alpha_n/\sqrt{n} \rightarrow \infty$ implies that $\sigma \in \Omega$ for a.a.e. σ , and so for any $U = U(G)$, $\mathbb{P}(\sigma_U, G) = (1 + o(1))\mathbb{P}(\sigma_U, \sigma \in \Omega, G)$ a.a.s; therefore,

$$\begin{aligned} \mathbb{P}(\sigma_U, G) &= (1 + o(1))\mathbb{P}(\sigma_U, \sigma \in \Omega, G) \\ &= (1 + o(1)) \sum_{\tau \in \Omega_U(\sigma)} \mathbb{P}(\tau, G) \\ &= (1 + o(1))K(G) \sum_{\tau \in \Omega_U(\sigma)} Q_{A \cup B, A \cup B}(G, \tau)Q_{B \cup C, C}(G, \tau) \end{aligned} \quad (6)$$

for a.a.e. G and σ . (Note that the $o(1)$ term can be taken to depend only on G , so there is no problem in pulling it out of the sum.) Applying (6) twice,

$$\begin{aligned} \mathbb{P}(\sigma_A | \sigma_B, G) &= \frac{\mathbb{P}(\sigma_{A \cup B}, G)}{\mathbb{P}(\sigma_B, G)} \\ &= (1 + o(1)) \frac{\sum_{\tau \in \Omega_{A \cup B}} Q_{A \cup B, A \cup B}(G, \tau)Q_{B \cup C, C}(G, \tau)}{\sum_{\tau \in \Omega_B} Q_{A \cup B, A \cup B}(G, \tau)Q_{B \cup C, C}(G, \tau)}. \end{aligned} \quad (7)$$

Note that $Q_{U_1, U_2}(\tau)$ depends on τ only through $\tau_{U_1 \cup U_2}$. In particular, in the numerator of (7), $Q_{A \cup B, A \cup B}(G, \tau)$ doesn't depend on τ since we only sum over τ with $\tau_{A \cup B} = \sigma_{A \cup B}$. Hence, the right hand side of (7) is just

$$(1 + o(1)) \frac{Q_{A \cup B, A \cup B}(G, \sigma) \sum_{\tau \in \Omega_{A \cup B}} Q_{B \cup C, C}(G, \tau)}{\left(\sum_{\tau \in \Omega_{B \cup C}} Q_{A \cup B, A \cup B}(G, \tau) \right) \left(\sum_{\tau \in \Omega_{A \cup B}} Q_{B \cup C, C}(G, \tau) \right)}, \quad (8)$$

where we could factorize the denominator because with τ_B fixed, $Q_{A \cup B, A \cup B}$ depends only on τ_A , while $Q_{B \cup C, C}$ depends only on τ_C . Cancelling the common terms, then multiplying top and bottom by $Q_{B \cup C, C}(G, \sigma)$, we have

$$\begin{aligned} (8) &= (1 + o(1)) \frac{Q_{A \cup B, A \cup B}(G, \sigma)}{\sum_{\tau \in \Omega_{B \cup C}} Q_{A \cup B, A \cup B}(G, \tau)} \\ &= (1 + o(1)) \frac{Q_{A \cup B, A \cup B}(G, \sigma) Q_{B \cup C, C}(G, \sigma)}{\sum_{\tau \in \Omega_{B \cup C}} Q_{A \cup B, A \cup B}(G, \tau) Q_{B \cup C, C}(G, \tau)} \\ &= (1 + o(1)) \frac{\mathbb{P}(G, \sigma)}{\mathbb{P}(G, \sigma_{B \cup C})} \\ &= (1 + o(1)) \mathbb{P}(\sigma_A | \sigma_{B \cup C}, G) \text{ a.a.s.} \end{aligned}$$

where the penultimate line used (6) for the denominator and (5) (plus the fact that $\sigma \in \Omega$ a.a.s.) for the numerator. On the other hand, recall from (7) that $(8) = (1 + o(1)) \mathbb{P}(\sigma_A | \sigma_B, G)$ a.a.s. □

Proof of Theorem 2.1. By the monotonicity of conditional variances,

$$\text{Var}(\sigma_\rho | G, \sigma_v, \sigma_{\partial G_R}) \leq \text{Var}(\sigma_\rho | G, \sigma_v).$$

Since $|G_R| = o(\sqrt{n})$ a.a.s. and $v \notin G_R$ a.a.s, it follows from Lemma 5.7 that σ_v and σ_ρ are a.a.s. conditionally independent given $\sigma_{\partial G_R}$ and G . Thus, $\text{Var}(\sigma_\rho | G, \sigma_v, \sigma_{\partial G_R}) \rightarrow \text{Var}(\sigma_\rho | G, \sigma_{\partial G_R})$. Now Proposition 5.2 implies that $|\text{Var}(\sigma_\rho | G, \sigma_{\partial G_R}) - \text{Var}(\tau_\rho | T, \tau_{\partial T_R})| \rightarrow 0$, but Theorem 5.1 says that

$$\text{Var}(\tau_\rho | T, \tau_{\partial T_R}) \rightarrow 1 \text{ a.a.s.}$$

and so $\text{Var}(\sigma_\rho | G, \sigma_{\partial G_R}) \rightarrow 1$ a.a.s. also. □

6 Parameter Estimation

We now prove Proposition 2.3

Proof. We first show that Theorem 3.1 gives us an estimator for a and b that is consistent when $(a - b)^2 > 2(a + b)$. First of all, we can estimate $d := (a + b)/2$ consistently by simply counting the number of edges. Thus, if we can estimate $f := (a - b)/2$ consistently then we do the same for a and b . Our estimator for f is

$$\hat{f} = (2kX_k - \hat{d}^k)^{1/k},$$

where \hat{d} is some estimator with $\hat{d} \rightarrow d$ \mathbb{P} -a.a.s. and $k = k(n)$ increases to infinity slowly enough so that $k(n) = o(\log^{1/4} n)$ and $\hat{d}^k - d^k \rightarrow 0$ \mathbb{P} -a.a.s. Take $\sqrt{\frac{a+b}{2}} < \rho < \frac{a-b}{2} = f$; by Chebyshev's inequality, $2kX_k - d^k \in [f^k - \rho^k, f^k + \rho^k]$ \mathbb{P} -a.a.s. Since $k = k(n) \rightarrow \infty$, $\rho^k = o(f^k)$. Thus, $2kX_k - d^k = (1 + o(1))f^k$ \mathbb{P} -a.a.s. Since $\hat{d}^k - d^k \rightarrow 0$, $2kX_k - \hat{d}^k = (1 + o(1))f^k$ \mathbb{P} -a.a.s. and so \hat{f} is a consistent estimator for f .

We next apply the second half of Theorem 2.2 to show that no estimator can be consistent when $(a - b)^2 < 2(a + b)$. In fact, if \hat{a} and \hat{b} are estimators for a and b which converge in probability, then their limit when $(a - b)^2 < 2(a + b)$ depends only on $a + b$. To see this, let α, β be another choice of parameters with $(\alpha - \beta)^2 < 2(\alpha + \beta)$ and $\alpha + \beta = a + b$; Let $\mathbb{Q}_n = \mathcal{G}_n(\alpha, \beta)$; take a^* to be the in-probability limit of \hat{a} under \mathbb{P}_n and α^* to be its limit under \mathbb{Q}_n . For an arbitrary $\epsilon > 0$, let A_n be the event $|\hat{a} - a^*| > \epsilon$; thus, $\mathbb{P}_n(A_n) \rightarrow 0$. By Theorem 2.2, $\mathbb{P}'_n(A_n) \rightarrow 0$ also. Since $\alpha + \beta = a + b$, we can apply Theorem 2.2 to \mathbb{Q}_n , implying that $\mathbb{Q}_n(A_n) \rightarrow 0$ and so $\alpha^* = a^*$. That is, \hat{a} converges to the same limit under \mathbb{Q}_n and \mathbb{P}_n . \square

Acknowledgments A.S. would like to thank Christian Borgs for suggesting the problem and Lenka Zdeborová for useful discussions. Part of this work was done while A.S. was at Microsoft Research, Redmond. The authors would like to thank Lenka Zdeborová for comments on a draft of this work and Peter Bickel for suggesting to study parameter estimation.

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